Complex dynamics of photon entanglement in two-mode Jaynes-Cummings model

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Abstract. We study the dynamics of the photon entanglement, $E_{\mathrm{N}}(t)$, for the two-mode Jaynes-Cummings model in the few-photon case. The atomic transitions associated with the photons with different polarizations are assumed to be independent and, hence, the evolution of the "+"- and "−"-polarized photons is formally separable. However, due to the photons indistinguishability such interaction still leads to entanglement of initially disentangled states owing to the non-linear dependence of the characteristic frequencies on the photon population numbers. The time dependence of entanglement is the result of superimposing oscillations with incommensurate frequencies. Therefore, $E_{\mathrm{N}}(t)$ is a quasi-periodic function of time with the complex profile strongly depending on the number of photons.

PACS numbers: 03.67.Bg,03.65.Yz,32.90.+a

1. Introduction

The property of many-body quantum systems to be in entangled, or inseparable, states is the object of significant constant interest [1]. Entanglement is the essentially quantum feature signifying the formation of new entities, the complexes of particles. If one would try to separate an individual particle from such complex by performing a measurement one would inevitably affect the states of other particles in the complex even if the direct interaction is absent.

Because of their highly unusual, from the classical point of view, properties the entangled states attract attention from both purely scientific and application points of view. As a result, as a problem of special importance there is the problem of producing entangled states. For example, nowadays the most developed and widely used method of generating the entangled photons is the parametric down conversion [2, 3], which is based on the two-photon radiative decay of excited states. This method, however, suffers from intrinsic limitations — very low yield and rescaling the wavelength of the emitted photons [4, 5, 6]. Therefore, there is the constant search of alternative sources of entangled light, which motivates thorough investigating of the physics of entangling.

The analysis of entangling photons requires different approach comparing to that of canonical quantum mechanical systems, say, qubits. First, one has to take into account indistinguishability of the photons, which imposes the severe restrictions on available set of states. Second, the interaction of the quantized electromagnetic field with medium excitations is accompanied by the processes of absorption and re-emission, hence the description should naturally incorporate the fact that the number of particles is not a constant. These circumstances suggest that the most suitable framework for description of the photon entanglement is provided by the quantum field theory. Within this framework the entanglement is characterized using the single-particle density matrix (SPDM)

$$K_{kq}(t) = \left\langle a_k^\dagger(t) a_q(t) \right\rangle, \tag{1}$$

where k and q enumerate the modes of the field. For example, for the situation considered in the present paper k and q run over "+" and "−" polarizations.

Throughout the paper we incorporate the time dependence into the Heisenber representation of the field operators $a_k(t) = \exp(\mathrm{i}\mathcal{H}t) a_k \exp(-\mathrm{i}\mathcal{H}t)$ with $\mathcal{H}$ being the Hamiltonian of the system, therefore, the average in formulas similar to (1) is taken with respect to the initial state $\langle \ldots \rangle = \langle \psi(t=0)| \ldots |\psi(t=0)\rangle$. We assume that initially the system is in a pure state and that the decohering processes are absent, therefore entanglement manifests itself in mixed single-particle states, that is the SPDM has rank (or the Schmidt number) larger than one [7, 8]. More specifically the entanglement can be quantified by the von Neumann entropy of the SPDM

$$E_{\mathrm{N}} = -\sum_i \widetilde{\rho}_i \log(\widetilde{\rho}_i), \tag{2}$$

where $\widetilde{\rho}_i$ are the normalized eigenvalues of the SPDM, so that $\sum_i \widetilde{\rho}_i \equiv 1$.

In the present paper we study the time dependence of entanglement of initially disentangled few-photon states in the two-mode Jaynes-Cummings model. This model closely corresponds to the dynamics of entanglement of the photons in the initially pumped cavity with a single atom admitting transitions with different helicities. The absorption and re-emission of photons with different polarizations is assumed to be

completely independent. From the canonical point of view this may seem to be similar to a many-body system without interaction, where entanglement is an integral of motion, that is initially disentangled states remain disentangled. However, as we will show, due to the complex character of the photons states the interaction with the atom leads to entangling states with different polarizations with non-trivial time dependence of $E_{\rm N}(t)$.

2. Photon entanglement in the context of Schwinger's model

The most convenient framework for the description of the photon entanglement is provided by Schwinger's model of angular momentum [9]. The components of the operator of angular momentum are defined in terms of the photons creation and annihilation operators as

$$\begin{aligned} \mathcal{J}_x &= \frac{1}{2}(a_+^\dagger a_- + a_-^\dagger a_+), \\ \mathcal{J}_y &= \frac{1}{2i}(a_+^\dagger a_- - a_-^\dagger a_+), \\ \mathcal{J}_z &= \frac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-). \end{aligned} \tag{3}$$

Using these operators one can characterize the photons states by the total angular momentum j and its projection m instead of the population numbers, n_+ and n_-, of the states with "+"- and "−"-polarizations. So, one has $|n_+, n_-\rangle = |j, m\rangle_{\rm S}$ with the relations $j = (n_+ + n_-)/2$ and $m = (n_+ - n_-)/2$.

Expressing the SPDM in terms of the mean values of the operator of the angular momentum one finds that the normalized eigenvalues of the SPDM can be expressed as

$$\widetilde{\rho}_{1,2} = \frac{1}{2}(1 - \widetilde{J}), \tag{4}$$

where $\widetilde{J} = |\langle \mathbf{J} \rangle|/j = 2|\langle \mathbf{J} \rangle|/N$ with $N = \mathrm{Tr}[K]$ being the total number of the photons. Thus, the entanglement can be written as $E_{\rm N} = F(\widetilde{J})$, where

$$F(x) = \frac{1}{2}\sum_{n=1,2}[1 + (-1)^n x]\log_2[1 + (-1)^n x]\,. \tag{5}$$

In particular, disentangled states are the states with the maximum magnitude of the average angular momentum $|\langle \mathbf{J} \rangle| = j$ and completely entangled ones are those with $|\langle \mathbf{J} \rangle| = 0$. It is interesting to compare this representation with the description of entanglement in the canonical quantum mechanical setup. Considering the two-photon case, $N = 2$, one can find that the total value of the angular momentum is directly related to the concurrence [10] $|C|^2 = 1 - \widetilde{J}^2$.

Such description of entanglement allows a simple parametrization of disentangled states. This is especially convenient for the problem with initially disentangled photons, which naturally appears when one considers the free dynamics of initially pumped cavity with an atom. The parametrization is based on the facts that for the disentangled states and the disentangled states only one has $|\langle \mathbf{J} \rangle| = j$ and that the average angular momentum $\langle \mathbf{J} \rangle$ transforms under rotations as a three-dimensional vector. Hence, any disentangled state can be turned by the rotations into $|j, j\rangle_{\rm S}$, or, in other words, any disentangled state can be presented as

$$|\psi(\beta_1, \beta_2, \beta_3)\rangle = \exp(-\mathrm{i}\mathcal{J}_z\beta_1)\exp(-\mathrm{i}\mathcal{J}_y\beta_2)\exp(-\mathrm{i}\mathcal{J}_z\beta_3)\,|j, j\rangle_{\rm S}\,, \tag{6}$$

where $\beta_{1,2,3}$ are the Euler angles. For the case of our main interest in the present paper only rotations by angle β_2 produce the states with different dynamics of entanglement, therefore, in the following consideration we limit our attention to $|\psi(\beta)\rangle = \exp(-\mathrm{i}\mathcal{J}_y\beta)\,|j,j\rangle_{\mathrm{S}}$.

3. Time dependence of entanglement

The dynamics of the two-mode Jaynes-Cummings model [11] is governed by the Hamiltonian, which we write down in terms of the creation and annihilation operators

$$\mathcal{H} = \sum_k \epsilon_k^{(\mathrm{p})} a_k^\dagger a_k + \sum_\kappa \epsilon_\kappa^{(\mathrm{e})} c_\kappa^\dagger c_\kappa + \mathcal{H}_+ + \mathcal{H}_-. \tag{7}$$

Here the first two terms describe the dynamics of the free field and the free atom, respectively. Here k runs over the photon polarizations, $+$ and $-$, and κ takes the values from the set of the electron states $\{\mathrm{g}\uparrow, \mathrm{g}\downarrow, \mathrm{e}\uparrow, \mathrm{e}\downarrow\}$ where g and e stand for the ground and excited atom levels and $\uparrow$, $\downarrow$ are the electron spins.

The interaction between the atom and photons preserves the helicity, that is the excitation of the electron state with the spin down at the ground level, $|\mathrm{g}\downarrow\rangle$, into the spin up state at the excited level $|\mathrm{e}\uparrow\rangle$ occurs through the absorption of "+"-polarized photon and so on. The interaction is described by the Hamiltonians $\mathcal{H}_k = \omega_k a_k \sigma_k^\dagger + \mathrm{h.c.}$, where $\sigma_+^\dagger = c_{\mathrm{e}\uparrow}^\dagger c_{\mathrm{g}\downarrow}$, $\sigma_-^\dagger = c_{\mathrm{e}\downarrow}^\dagger c_{\mathrm{g}\uparrow}$ and $\omega_\pm$ are the respective Rabi frequencies.

The interplay between different characteristic frequencies determining the dynamics of the system leads to the complex time dependence of the amplitudes. In order to concentrate on the main features, we adopt the resonant approximation and set $\epsilon_\kappa^{(\mathrm{e})} = \epsilon_k^{(\mathrm{p})} = 0$. Additionally we assume that the symmetry between transitions with different helicities is not broken so that $\omega_+ = \omega_- = \omega_{\mathrm{R}}$.

In order to describe the time dependence of entanglement we use the explicit form of the Heisenberg representation for the photon operators $a_k(t) = \exp(\mathrm{i}t\mathcal{H}_k)a_k\exp(-\mathrm{i}t\mathcal{H}_k)$. Taking into account the separability of the dynamics we have [12]

$$a_k(t) = e^{i\mathcal{C}_k t}\left\{\left[\cos\left(\bar{\mathcal{C}}_k t\right) - \mathrm{i}\mathcal{C}_k\frac{\sin\left(\bar{\mathcal{C}}_k t\right)}{\bar{\mathcal{C}}_k}\right]a_k - \mathrm{i}\frac{\sin\left(\bar{\mathcal{C}}_k t\right)}{\bar{\mathcal{C}}_k}\sigma_k\right\}, \tag{8}$$

where $\mathcal{C}_k^2 = \omega_{\mathrm{R}} a_k^\dagger a_k$ and $\bar{\mathcal{C}}_k^2 = \omega_{\mathrm{R}}(a_k^\dagger a_k + 1)$. Taking into account that initially the atom is not excited the initial state of the system is presented as

$$|\Psi(0)\rangle = |\psi(\beta)\rangle\,|0\rangle_e\,, \tag{9}$$

where $|\psi(\beta)\rangle$ is the disentangled photon state obtained by rotations from the state $|j,j\rangle_{\mathrm{S}}$ [see (6)] and $|0\rangle_e$ denotes the state of the atom with both electrons at the ground level. It should be noted with this regard that the second term in (8) does not contribute to $K_{kq}(t)$ because it vanishes while acting on the initial state of the atom.

First, we consider $\langle\mathcal{J}_z(t)\rangle$. Taking into account the identity $f\left(\bar{\mathcal{C}}_k\right)a_k = a_k f\left(\mathcal{C}_k\right)$ we find

$$\langle\mathcal{J}_z(t)\rangle = \langle\mathcal{J}_z\rangle + \frac{1}{4}\left\langle[\cos(2\mathcal{C}_+ t) - \cos(2\mathcal{C}_- t)]\right\rangle. \tag{10}$$

Here and below the average is taken over the initial photon state $\langle\ldots\rangle = \langle\psi(\beta)|\ldots|\psi(\beta)\rangle$. The second term in (10) has the maximum magnitude in completely

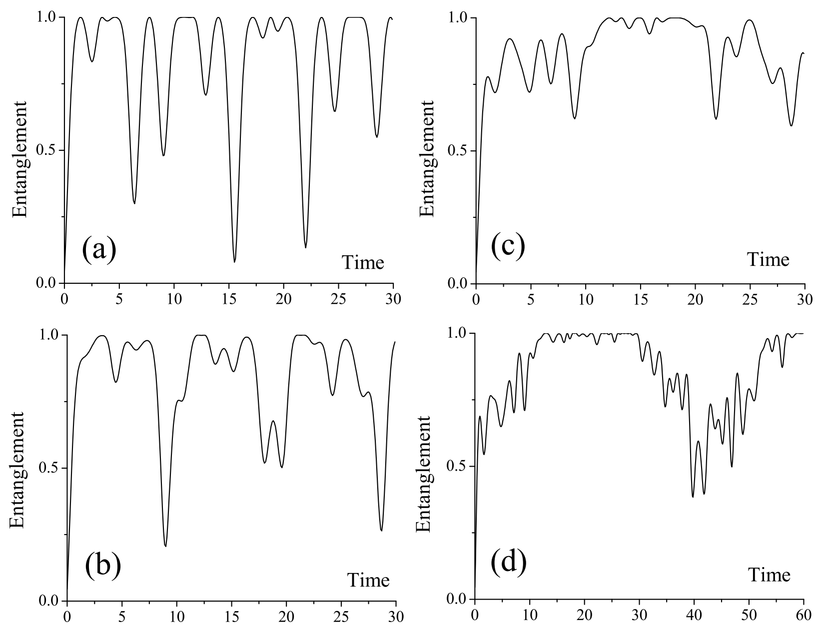

Figure 1. Time dependence of entanglement of the initially disentangled N photons: (a) $N = 2$, (b) $N = 3$, (c) $N = 4$ and (d) $N = 5$ (notice the different total duration). The time is measured in $1/\omega_{\mathrm{R}}$.

polarized case, that is when $\langle \mathcal{J}_z \rangle = j$, when it has the form of oscillations with the frequency $2N\omega_{\mathrm{R}}$ due to absorption and re-emission of the single photon by the atom signifying the entanglement of photons and atomic states. The magnitude of this term monotonously decreases with β and vanishes identically in the unpolarized case, when $\langle \mathcal{J}_z \rangle = 0$.

It should be noted that even in the polarized case the relative contribution of the last term in (10) is relatively small $\propto 1/N$ and thus one can consider $\langle \mathcal{J}_z \rangle$ as imposing the limitation from above on the maximum value of achievable entanglement. Therefore we concentrate on the unpolarized case when $\langle \mathcal{J}_z(t) \rangle \equiv 0$ and the time dependence of entanglement is only due to the transversal component of the angular momentum $|\langle \mathbf{J}(t) \rangle|^2 = \langle \mathcal{J}_x(t) \rangle^2 + \langle \mathcal{J}_y(t) \rangle^2$.

We present in figure 1 the results of numerical evaluation of entanglement time dependence calculated according to equation (5). First of all we would like to emphasize that the entanglement reaches the maximum value even in the case when $N > 2$, when the sole entanglement with the atomic state is not sufficient to support $E_{\mathrm{N}} = 1$. In order to provide a qualitative explanation of the origin of entanglement let us consider a two-photon state. In the basis of population numbers any such state has the form

$$|\psi\rangle = \alpha_{2,0}\,|2,0\rangle + \alpha_{1,1}\,|1,1\rangle + \alpha_{0,2}\,|0,2\rangle\,, \tag{11}$$

where α_{n_+,n_-} are the respective time-dependent amplitudes. Using this representation we find for the concurrence

$$C = 2\alpha_{2,0}\alpha_{0,2} - \alpha_{1,1}^2. \tag{12}$$

The initial condition is determined by the requirement for the state to be disentangled, thus $C(t = 0) = 0$. For a free field the characteristic frequencies of the amplitudes α_{n_+,n_-} are *linearly* proportional to the total number of photons and, as a result, the time dependence factors out in (12) yielding $C(t) = 0$, that is the state remains disentangled. However, as can be seen from Heisenberg representation (8), the interaction with the atom essentially modifies the dependence of the characteristic frequencies on the number of photons, which becomes $\propto \sqrt{N}$. Thus the time dependence of the first and the second terms in (12) is determined by the incommensurate frequencies $\Omega_1/\Omega_2 \sim \sqrt{2}$, which inevitably leads to the phase desynchronizing and to the violation of the condition $C = 0$.

The second important consequence of the involvement of incommensurate frequencies into the dynamics is the complex (quasi-periodic) profile of $E_{\mathrm{N}}(t)$, which dominates the time dependence of entanglement after the initial regular raise with the typical time scale $\sim 1/\omega_{\mathrm{R}}$. This makes $E_{\mathrm{N}}(t)$ to some extent unpredictable and imposes strong requirements with regard to the control of the life time of the photons inside the cavity and to the initial conditions. The addition of a single photon drastically changes the value of entanglement at any particular instant beyond the period of initial raise. It should be noted, however, that when the number of photons in the cavity increases the contribution of particular frequencies becomes less important and the profile of $E_{\mathrm{N}}(t)$ changes toward some general regular shape. The initial stage of the transformation can be seen by comparing figures 1c and 1d. The detailed analysis of the regular profile in the limit $N \gg 1$ will be provided elsewhere. Here we would like to note that the normalized value of the average angular momentum can be presented in this limit as a periodic sequence of Gaussian bumps with the characteristic period $\sim N^{3/2}/\omega_{\mathrm{R}}$ and the typical entanglement time $\sim N/\omega_{\mathrm{R}}$.

4. Conclusion

We have considered the time dependence of entanglement of initially disentangled few-photons states within the two-mode Jaynes-Cummings model. The processes of absorption and re-emission of photons with different polarizations are assumed independent and, therefore, the dynamics of "+"- and "−"-polarized photons is completely separable. Applying the standard description of entanglement straightforwardly this may seem to imply that the states with different polarizations will stay disentangled. However, the indistinguishability of the photons leads to the necessity to apply the standard picture with the special care. Indeed, except in the simplest case, when all photons have the same polarization, each photon is always in the superposition of the states and, therefore, is always affected by "both parts" of the dynamics. As a result, in order to make a conclusion regarding entanglement one has to consider closely the evolution of the relation between the amplitudes.

We have studied the photon entanglement using the formalism of the single-particle density matrix. We have established the relation between the entanglement and the magnitude of the average angular momentum defined with the help of Schwinger's model. Using this relation and the exact Heisenberg representation of the photon creation and annihilation operators we have calculated the time dependence

of entanglement $E_{\mathrm{N}}(t)$. We have shown that $E_{\mathrm{N}}(t)$ has two important features. The interaction with the single atom may lead to complete entanglement $E_{\mathrm{N}} = 1$. This is related to the fact that the characteristic frequencies determining the time evolution of the photon amplitudes depend *non-linearly* on the population numbers. In turn, the superposition of oscillations with incommensurate frequencies results in quasi-periodic $E_{\mathrm{N}}(t)$ with complex profile, especially in the few-photon case $N \sim 1$. The specific form of the profile drastically depends on the number of particles and transforms into a regular pattern in the limit $N \gg 1$,

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